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A Theorem Connected with Irrational Numbers.

By William Duncan MacMillan.

Let us imagine in that portion of the plane in which the coordinates are both positive that the points whose coordinates are both integers are marked with a dot (lattice points), and let us imagine a straight line drawn through the origin making an angle ϕ with the x-axis. Let $\beta = \tan \phi$, $0 < \phi < \pi/2$. Consider the lattice points in the vertical line whose x-coordinates is q_k. One of these points is closer to the line ϕ than any other point in the same vertical line. Let the y-coordinate of this point be p_k, and let y_k be the ordinate of the line ϕ corresponding to the abscissa q_k, so that $y_k = \beta q_k$. We will call $p_k - y_k = p_k - \beta q_k = d_k$ the distance of the point q_k, p_k from the line ϕ, though of course it is not the perpendicular distance. From the definition it is clear that d_k is positive if the lattice point lies above the line ϕ, and negative if it lies below the line ϕ.

The theorem which is proved in the following pages relates to the limiting value of the geometric mean of the $|d_k|$. If G_n denotes the geometric mean of the first n quantities $|d_k|$, and if β is an irrational number which satisfies a certain condition, then the limit of G_n, as n approaches infinity, is $1/(2e)$, where e is the logarithmic base. If β is rational the limit of G_n is, of course, zero. For values of β other than the two classes already mentioned G_n can not be said to have a limit as it oscillates between $1/(2e)$ and l_β, where $0 \overset{=}{<} l_\beta < 1/(2e)$.

As the method of proof is based upon the properties of simple continued fractions, we shall set forth explicitly those formulas which will prove most useful. Let us suppose β is developed as a simple continued fraction

$$\beta = a_0 + \frac{1}{a_1 +}\, \frac{1}{a_2 +} \cdots,$$

and let p_n/q_n be the n-th principal convergent, and $p_n^{(s)}/q_n^{(s)}$ be an intermediate convergent. Then from the theory of continued fractions we will have

$$\begin{aligned} p_{n+1} &= a_{n+1} p_n + p_{n-1}, & \varepsilon_n &= p_n - \beta q_n, \\ q_{n+1} &= a_{n+1} q_n + q_{n-1}, & |\varepsilon_{n-1}| &> |\varepsilon_n| > |\varepsilon_{n+1}|, \\ \varepsilon_{n+1} &= a_{n+1} \varepsilon_n + \varepsilon_{n-1}, \end{aligned}$$

$$p_n^{(s)} = (s+1) p_n + p_{n-1}, \quad q_n^{(s)} = (s+1) q_n + q_{n-1}, \quad \varepsilon_n^{(s)} = (s+1) \varepsilon_n + \varepsilon_{n-1}.$$

Since the principal convergent points q_n, p_n are the closest approximations to β it follows that these points lie closest to the line ϕ. Since the ε_n are alternately positive and negative, these principal convergent points lie alternately above and below the line ϕ. If the convergent points q_{n-1}, p_{n-1} and q_{n+1}, p_{n+1} be joined by a straight line, the intermediate convergent points $q_n^{(s)}$, $p_n^{(s)}$ will lie on this straight line, and between this straight line and the line ϕ there are no lattice points. Finally,

$$\frac{1}{q_{n+1}} > |\varepsilon_n| > \frac{a_{n+2}}{q_{n+2}},$$

or, simpler,

$$\frac{1}{a_{n+1}q_n} > |\varepsilon_n| > \frac{1}{(a_{n+1}+2)\,q_n}.$$

Let us join the origin and the convergent point q_n, p_n by a straight line and denote the distances of the lattice points from this straight line by δ_k. In order to avoid confusion we shall call this straight line the rational line. It is clear that the distances δ_k in some order have the values

$$+\frac{1}{q_n},\ -\frac{1}{q_n},\ +\frac{2}{q_n},\ -\frac{2}{q_n},\ \ldots,\ \pm\frac{(q_n-1)/2}{q_n} \text{ or } \pm 1/2,$$

according as q_n is odd or even, for there are q_n-1 such distances (excluding the end points). If we suppose two of them are equal, say δ_k and δ_j, then we have $\delta_j-\delta_k=0$; that is,

$$(p_j-p_k)-\frac{p_n}{q_n}(q_j-q_k)=0,$$

whence

$$\frac{p_j-p_k}{q_j-q_k}=\frac{p_n}{q_n},$$

which is impossible since $|p_j-p_k|<p_n$, $|q_j-q_k|<q_n$, and p_n and q_n are relatively prime. Furthermore, from symmetry we have $\delta_k=-\delta_{n-k}$.

In computing the geometric mean of the quantities $|\delta_k|$ for the rational line we have

$$\prod_{k=1}^{q_n-1}|\delta_k|=\frac{1}{q_n}\cdot\frac{2}{q_n}\cdot\frac{3}{q_n}\cdot\ldots\ldots\cdot\frac{m}{q_n}\cdot\frac{m}{q_n}\cdot\ldots\ldots\cdot\frac{2}{q_n}\cdot\frac{1}{q_n} \quad\text{if } q_n=2m+1,$$

$$\prod_{k=1}^{q_n-1}|\delta_k|=\frac{1}{q_n}\cdot\frac{2}{q_n}\cdot\ldots\ldots\cdot\frac{m}{q_n}\cdot\frac{1}{2}\cdot\frac{m}{q_n}\cdot\ldots\ldots\cdot\frac{2}{q_n}\cdot\frac{1}{q_n} \quad\text{if } q_n=2m+2.$$

In general, the factors will not occur in the order above written, but all of the above factors occur with none others. It will be observed that the distance for the end point δ_n has been omitted since it is zero. It is seen, therefore, that we are computing the geometric mean out to the n-th convergent point, but that this point itself is excluded. We have then

$$\left[\prod_{k=1}^{q_n-1}|\delta_k|\right]^{\frac{1}{q_n-1}}=\left[\frac{(m!)^2}{(2m+1)^{2m}}\right]^{\frac{1}{2m}}=\left[\frac{m!}{(2m+1)^m}\right]^{\frac{1}{m}} \text{ if } q_n=2m+1,$$

$$\left[\prod_{k=1}^{q_n-1}|\delta_k|\right]^{\frac{1}{q_n-1}}=\left[\frac{1}{2}\frac{(m!)^2}{(2m+2)^{2m}}\right]^{\frac{1}{2m+1}}=\frac{1}{2}\left[\frac{m!}{(m+1)^m}\right]^{\frac{2}{2m+1}} \text{ if } q_n=2m+2.$$

In order to show that these expressions have definite limiting values as n, and consequently m, increases indefinitely we will use the theorem of Cauchy*

$$\underset{n=\infty}{L}\,[f(n)]^{\frac{1}{n}}=\underset{n=\infty}{L}\left[\frac{f(n+1)}{f(n)}\right].$$

If $q_n=2m+1$, we have $f(m)=\dfrac{m!}{(2m+1)^m}$. Consequently,

$$\frac{f(m+1)}{f(m)}=\frac{m+1}{2m+3}\left(\frac{2m+1}{2m+3}\right)^m=\frac{m+1}{2m+3}\left(1-\frac{1}{m+3/2}\right)^m,$$

and the limit as m increases indefinitely is $1/(2e)$, where $e=2\cdot71828\ldots.$, the Naperian base. In precisely the same fashion it is found that $\dfrac{1}{2}\left[\dfrac{m!}{(m+1)^m}\right]^{\frac{2}{2m+1}}$ also has the limit $1/(2e)$. It follows then that whether q_n is even or odd

$$\underset{n=\infty}{L}\left[\prod_{k=1}^{q_n-1}|\delta_k|\right]^{\frac{1}{q_n-1}}=\frac{1}{2e}.$$

Consider now the geometric mean for the irrational line. Let

$$G_s=\left[\prod_{k=1}^{s}|d_k|\right]^{\frac{1}{s}},$$

where $d_k=p_k-\beta q_k$ are the distances from the irrational line corresponding to the δ_k for the rational line.

If every $|d_k|$ is replaced by $|\delta_k|+\dfrac{1}{q_n}$, the resulting geometric mean will be too great, since

$$|d_k|<|\delta_k|+|\varepsilon_n|<|\delta_k|+\frac{1}{q_n};$$

and if every $|d_k|$ is replaced by $|\delta_k|-\dfrac{1}{q_n}$, except for the two points for which this expression is zero, the resulting geometric mean will be too small. The two exceptional points for which $|\delta_k| = \frac{1}{q_n}$ are the points (q_{n-1}, p_{n-1}) and the symmetrical point $((a_n-1)q_{n-1}+q_{n-2}, (a_n-1)p_{n-1}+p_{n-2})$. These two points are at equal distances from the ends of the rational line and lie on opposite sides of it. When the rational line is rotated into coincidence with the irrational line, the distance of one of these points is increased while the other is decreased. It is readily verified that it is the distance of the point q_{n-1}, p_{n-1} which is decreased. From the geometry it follows that the amount of this decrease is $\frac{q_{n-1}}{q_n}|\varepsilon_n| < \frac{1}{2q_n}$. Hence $|d_{q_{n-1}}| > \frac{1}{2q_n}$, and this is the only distance which is less than $1/q_n$. It results, then, that

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* "Analyse Algebrique." See also Chrystal's "Algebra," II, p. 84.

$$\left[\frac{\frac{1}{2}(m-1)\,!}{(2m+1)^m}\right]^{\frac{1}{m}} < G_{2m} < \left[\frac{(m+1)\,!}{(2m+1)^m}\right]^{\frac{1}{m}}, \text{ if } q_n=2m+1,$$

and

$$\frac{1}{2}\left[\frac{m}{4}\cdot\frac{(m-1)\,!^2}{(m+1)^{2m+1}}\right]^{\frac{1}{2m+1}} < G_{2m+1} < \frac{1}{2}\left[(m+2)\frac{(m+1)\,!^2}{(m+1)^{2m+1}}\right]^{\frac{1}{2m+1}}, \text{ if } q_n=2m+2.$$

The limit of all of these functions of m is $1/(2e)$. It follows, therefore, that *the limit of the geometric mean of the quantities* $|d_k|$ *taken out to, but not including, a principal convergent point is* $1/(2e)$ *for every irrational number.*

We wish, however, to include the distance of the convergent point itself. For this point $|d_{q_n}| = |\varepsilon_n|$ and

$$\frac{1}{(a_{n+1}+2)\,q_n} < |\varepsilon_n| < \frac{1}{a_{n+1}\,q_n}.$$

Since the limit of $\left(\frac{1}{q_n}\right)^{\frac{1}{q_n}}$ for large values of q_n is unity, we have

$$\underset{n=\infty}{L}\ |\varepsilon_n|^{\frac{1}{q_n}} = \underset{n=\infty}{L}\left(\frac{1}{a_{n+1}}\right)^{\frac{1}{q_n}}.$$

The value of q_n depends upon $a_1, \ldots, a_n$, but not upon a_{n+1}. Since a_{n+1} is an integer, $\left(\frac{1}{a_{n+1}}\right)^{\frac{1}{q_n}}$ lies between zero and unity, and if it has a limit as n increases indefinitely, say l_β, then it is clear that

$$\underset{n=\infty}{L}\ G_{q_n} = (l_\beta)/(2e),$$

and, in particular, if $l_\beta=1$ the limit, not only of G_{q_n-1} but also of G_{q_n}, is $1/(2e)$. The limit $l_\beta=1$ will be assured if

$$a_{n+1} \leqq M\, q_n(q_n+1) \ldots . (q_n+s)$$

or, its equivalent,

$$a_{n+1} \leqq M q_n^{s+1}$$

for every n greater than a fixed n_1, where M is a chosen positive quantity and s is any assigned positive integer. This condition is satisfied by every real algebraic number.* It is also satisfied by certain transcendental numbers, for example, the irrational number e, for which the values of the a_n are known. Since $q_n > a_1 \cdot a_2 \cdot \ldots . \cdot a_n$ it is seen that q_n increases very rapidly even for moderate values of the a_j; and, therefore, that the condition is satisfied by a very wide class of irrationals. In order that the condition be not satisfied it is necessary that ultimately the a_n shall increase with exceeding rapidity.

If the quantity $\left(\frac{1}{a_{n+1}}\right)^{\frac{1}{q_n}}$ does not have any fixed limit as n increases indefinitely but oscillates between zero and unity, then G_{q_n} oscillates between zero and $1/(2e)$ although G_{q_n-1} has the fixed limit $1/(2e)$.

In what follows we shall assume that the condition $\underset{n=\infty}{L}\left(\frac{1}{a_{n+1}}\right)^{\frac{1}{q_n}} = 1$ is satisfied, and free ourselves of the restriction that we proceed to the limit along a particular set of points. Let us take $b_n < a_{n+1}$ and consider the geometric mean out to the point $b_n q_n$. The distance of this point from the irrational line is $b_n |\varepsilon_n|$. Since $a_{n+1}\varepsilon_n = \varepsilon_{n+1} - \varepsilon_{n-1}$ and $|\varepsilon_{n-1}| < \frac{1}{q_n}$ we have $b_n|\varepsilon_n| < \frac{b_n}{a_{n+1}} \cdot \frac{1}{q_n} < \frac{1}{q_n}$. We draw now the rational line from the origin to the point $b_n q_n$, $b_n p_n$. For this rational line the $|\delta_k|$ in some order have the values $0, \frac{1}{q_n}, \frac{2}{q_n}, \ldots, \frac{m}{q_n}$ or $\frac{1}{2}$ according as q_n is odd or even. The points for which $|\delta_k| = 0$ are the integral multiples of q_n; and the points for which $|\delta_k| = 1/q_n$ are the points whose abscissæ are $sq_n \pm q_{n-1}$, $s = 1, \ldots, b_n$. When the rational line is rotated into coincidence with the irrational line, the $|\delta_k|$ for the points $sq_n - q_{n-1}$ are increased, while for the points $sq_n + q_{n-1}$ they are decreased. It is, therefore, the points $sq_n + q_{n-1}$, the intermediate convergent points, which engage our attention. We have

$$\text{for } k = sq_n, \qquad |d_k| = 0 + s|\varepsilon_n|, \qquad s \leqq b_n,$$

$$\text{for } k = sq_n + q_{n-1}, \qquad |d_k| = \frac{1}{q_n} - \left(s + \frac{q_{n-1}}{q_n}\right)|\varepsilon_n|, \qquad s \leqq b_n - 1.$$

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* Liouville *Journal de Mathématiques*, XVI (1851).

Since $|\varepsilon_n| < \dfrac{1}{q_{n+1}}$ we have also

$$\left(s+\frac{q_{n-1}}{q_n}\right)|\varepsilon_n| < \left(s+\frac{q_{n-1}}{q_n}\right)\frac{1}{a_{n+1}q_n+q_{n-1}} = \frac{1}{q_n}\,\frac{s+\dfrac{q_{n-1}}{q_n}}{a_{n+1}+\dfrac{q_{n-1}}{q_n}},$$

so that

$$\frac{1}{q_n} - \left(s+\frac{q_{n-1}}{q_n}\right)|\varepsilon_n| > \frac{1}{q_n}\,\frac{a_{n+1}-s}{a_{n+1}+1},\quad s \leqq b_n - 1.$$

When the rational line is rotated into coincidence with the irrational line, no $|\delta_k|$ is altered by as much as $\dfrac{1}{q_n}$. If then all of the $|\delta_k| > \dfrac{1}{q_n}$ be diminished by $1/q_n$ we will obtain a geometric mean which is less than $G_{b_n q_n}$, and if all of the $|\delta_k|$ be increased by $\dfrac{1}{q_n}$, without exception, we will obtain a geometric mean which is greater than $G_{b_n q_n}$. Carrying out these operations first for q_n odd and second for q_n even, we find

$$\left[\left(\frac{|\varepsilon_n|}{a_{n+1}+1}\right)^{b_n}\frac{(m-1)\,!^{2b_n}}{q_n^{\,b_n(q_n-1)}}\right]^{\frac{1}{b_n q_n}} < G_{b_n q_n} < \left[\frac{(m+1)\,!^{2b_n}}{q_n^{\,b_n q_n}}\right]^{\frac{1}{b_n q_n}} \text{ if } q_n = 2m+1,$$

$$\left[\left(\frac{|\varepsilon_n|}{a_{n+1}+1}\right)^{b_n}\frac{m^{b_n}(m-1)\,!^{2b_n}}{q_n^{\,b_n(q_n-1)}}\right]^{\frac{1}{b_n q_n}} < G_{b_n q_n} < \left[\frac{(m+2)\,(m+1)\,!^{2b_n}}{q_n^{\,b_n q_n}}\right]^{\frac{1}{b_n q_n}} \text{ if } q_n = 2m+2.$$

These two inequalities can be replaced by the single one, whether q_n be even or odd,

$$\left(\frac{1}{a_{n+1}+2}\right)^{\frac{2}{q_n}}\left[\frac{(m-1)\,!^2}{q_n^{\,q_n}}\right]^{\frac{1}{q_n}} < G_{b_n q_n} < \left[\frac{(m+2)\,!^2}{q_n^{\,q_n}}\right]^{\frac{1}{q_n}}.$$

Since by hypothesis the limit of $\left(\dfrac{1}{a_{n+1}+2}\right)^{\frac{1}{q_n}} = 1$, both extremes of this inequality have the limiting value $1/(2e)$. It follows that the limit of $G_{b_n q_n}$ is $1/(2e)$.

The general point q can be written

$$q = b_n q_n + b_{n-1} q_{n-1} + \ldots . + b_1 q_1 + b_0,$$

where $b_k \leqq a_{k+1}$. This point, therefore, lies between $b_n q_n$ and $b_{n+1} q_n$, which is an interval of length $q_n = a_n q_{n-1} + q_{n-2}$. If now we attempt to continue with the interval $b_{n-1} q_{n-1}$, a difficulty is encountered at the intermediate convergent point $b_n q_n + q_{n-1}$. For values of b_n in the neighborhood of a_{n+1} the distance of this lattice point is of the order $|\varepsilon_n|$, while the root which we take is of the order $\dfrac{1}{q_{n-1}}$. Our hypothesis on the a_{n+1} would not cover the situation at this point. If, therefore, $b_{n-1}\neq 0$, it will be convenient to include at one step the interval q_n+q_{n-1} which immediately precedes this point; that is, the interval from $(b_n-1)q_n$ to $b_n q_n+q_{n-1}$, and we will suppose the straight line joining these two points to have been drawn.

Since p_n+p_{n-1} is prime to q_n+q_{n-1}, the values of the $|\delta_k|$ for this rational line are integral multiples of $\dfrac{1}{q_n+q_{n-1}}$, none vanishing except for the end points. Counting k from the beginning of this rational line we have $|\delta_k|=\dfrac{1}{q_n+q_{n-1}}$ for $k=q_n$, and $|\delta_k|=\dfrac{2}{q_n+q_{n-1}}$ for $k=q_n-q_{n-1}$. The $|\delta_k|$ for the symmetrical points have the same values.

Now $|b_n\varepsilon_n+\varepsilon_{n-1}|\leqq|\varepsilon_n+\varepsilon_{n-1}|<\dfrac{2}{q_n+q_{n-1}}$, so that when the rational line is moved into coincidence with the irrational line, no $|\delta_k|$ is altered by as much as $\dfrac{2}{q_n+q_{n-1}}$. We have, therefore, to examine the points for which $|\delta_k|$ equals $\dfrac{1}{q_n+q_{n-1}}$ or $\dfrac{2}{q_n+q_{n-1}}$. For the points for which $|\delta_k|=\dfrac{1}{q_n+q_{n-1}}$, $k=q_n$ and q_{n-1}. Hence $|d_k|=b_n|\varepsilon_n|$ and $|(b_n-1)\varepsilon_n+\varepsilon_{n-1}|$ respectively, both of which are greater than $|\varepsilon_n|$. For the points for which $|\delta_k|=\dfrac{2}{q_n+q_{n-1}}$, we have $k=q_n-q_{n-1}$ and $2q_{n-1}$. At these points $|d_k|$ is equal to $|b_n\varepsilon_n-\varepsilon_{n-1}|$ and $|(b_n-1)\varepsilon_n+2\varepsilon_{n-1}|$ and both of these quantities are greater than $\dfrac{1}{q_n+q_{n-1}}$. If, for convenience, of notation we denote q_n+q_{n-1} by Q_n and the geometric mean of the quantities $|d_k|$ between the points $(b_n-1)q_n+1$ and $b_n q_n+q_{n-1}$, both inclusive, by G_{Q_n}, then whether Q_n is even or odd we will have

$$Q_n=2m+1 \quad\text{or}\quad Q_n=2m+2,$$

$$\left[\frac{(m-2)\,!^2}{Q_n^{\,Q_n}\,a_{n+1}^2}\right]^{\frac{1}{Q_n}}<G_{Q_n}<\left[\frac{(m+3)\,!^2}{Q_n^{\,Q_n}}\right]^{\frac{1}{Q_n}},$$

and these expressions also have the limit $1/(2e)$.

No further difficulties are encountered in proceeding from the point $b_n q_n+q_{n-1}$ to the point $b_n q_n+b_{n-1}q_{n-1}$ since the minimum $|d_k|$ occur at the intermediate convergent points and have the values $|b_n\varepsilon_n+s\varepsilon_{n-1}|>|\varepsilon_{n-1}|$ and $|d_k|$ does not differ from $|\delta_k|$ by as much as $\dfrac{1}{q_{n-1}}$.

Let us suppose that, proceeding in this manner, we have arrived at the point

$$b_nq_n+b_{n-1}q_{n-1}+\dots\dots+b_{n-r}q_{n-r},$$

and we endeavor to continue to the point

$$b_nq_n+\dots\dots+b_{n-r}q_{n-r}+b_{n-r-1}q_{n-r-1}.$$

If b_{n-r-1} is zero the interval is of length less than q_{n-r-1} and we proceed to an interval which is a multiple of q_{n-r-2}, and in such a case no difficulties arise at the subintermediate convergent points. But if $b_{n-r}\neq 0$, then

$$b_nq_n+\dots\dots+b_{n-r}q_{n-r}+q_{n-r-1}$$

is a subintermediate convergent point in an interval of length q_{n-r}. To arrive at this point we start from the point $b_nq_n+\dots\dots+(b_{n-r}-1)q_{n-r}$ and include in one step the interval $(q_{n-r}+q_{n-r-1})$, which brings us to the point $b_nq_n+\dots\dots+b_{n-r}q_{n-r}+q_{n-r-1}$.

For notation let us take

$$P_{n-r}=p_{n-r}+p_{n-r-1},\quad Q_{n-r}=q_{n-r}+q_{n-r-1},\quad d_k^{(n-r)}=d_{b_nq_n+\dots\dots(b_n-1)q_{n-r}+k}.$$

Then, since P_{n-r} is prime to Q_{n-r} the $|\delta_k^{(n-r)}|$ have values which are integral multiples of $\frac{1}{Q_{n-r}}$. If k equals q_{n-r-1} or q_{n-r} then $|\delta_k^{(n-r)}|=\frac{1}{Q_{n-r}}$; and if k equals $2q_{n-r-1}$ or $q_{n-r}-q_{n-r-1}$, then $|\delta_k^{(n-r)}|=\frac{2}{Q_{n-r}}$. The difference between the rational line and the irrational line for this interval at no point exceeds $|b_n\varepsilon_n+\dots\dots+b_{n-r}\varepsilon_{n-r}+\varepsilon_{n-r-1}|\leqq|\varepsilon_{n-r}+\varepsilon_{n-r-1}|<\frac{2}{Q_{n-r}}$. Hence, when the rational line is moved into coincidence with the irrational line no $|\delta_k^{(n-r)}|$ is altered by as much as $\frac{2}{Q_{n-r}}$. We need, therefore, to examine those points for which $|\delta_k^{(n-r)}|<\frac{3}{Q_{n-r}}$. For the points for which $|\delta_k^{(n-r)}|=\frac{1}{Q_{n-r}}$ we have $|d_k^{(n-r)}|=|b_n\varepsilon_n+\dots\dots+(b_{n-r}-1)\varepsilon_{n-r}+\varepsilon_{n-r-1}|$ and $|b_n\varepsilon_n+\dots\dots+b_{n-r}\varepsilon_{n-r}|$ respectively, and both of these quantities are less than $|\varepsilon_{n-r}|$. For the points for which $|\delta_k^{(n-r)}|=\frac{2}{Q_{n-r}}$, we have

$$|b_n\varepsilon_n+\dots\dots+b_{n-r}\varepsilon_{n-r}+2\varepsilon_{n-r-1}|$$

and

$$|b_n\varepsilon_n+\dots\dots+b_{n-r}\varepsilon_{n-r}-\varepsilon_{n-r-1}|,$$

each of which is greater than $\frac{1}{Q_{n-r}}$.

If, therefore, $G_{Q_{n-r}}$ denotes the geometric mean of the quantities $|d_k^{(n-r)}|$, $k=1, \ldots, Q_{n-r}$, and if $Q_{n-r}=2m+1$ or $2m+2$ we will have

$$\left[\frac{(m-2)\,!^2}{Q_{n-r}^{Q_{n-r}-3}}\,|\varepsilon_{n-r}|^3\right]^{\frac{1}{Q_{n-r}}} < G_{Q_{n-r}} < \left[\frac{(m+3)\,!}{Q_{n-r}^{Q_{n-r}}}\right]^{\frac{1}{Q_{n-r}}}.$$

For a fixed r the two extremes of this inequality have the limit $1/(2e)$ as n increases indefinitely.

We can now proceed from the point $b_nq_n+\ldots+b_{n-r}q_{n-r}+q_{n-r-1}$ to the point $b_nq_n+\ldots+b_{n-r-1}q_{n-r-1}$ without further difficulty. Draw the rational line connecting these two points. The distance between this line and the irrational line is nowhere greater than $\frac{1}{q_{n-r-1}}$. The values of the $|d|$ for the points for which $|\delta|=0$ or 1 are in every case greater than $|\varepsilon_{n-r-1}|$. Consequently, inequalities similar to the above can be constructed, and these inequalities have the same limits as before.

We have been compelled to break up the interval $q=b_nq_n+\ldots+b_{n-r}q_{n-r}+\ldots+b_0$ into the intervals $q=c_nq_n+Q_n+\ldots+c_{n-r}q_{n-r}+Q_{n-r}+\ldots+c_0$, where $c_n=b_n-1$, $c_j=b_j-2$, $j=2, \ldots, n-1$, $c_1=b_1-1$, $c_0=b_0$. Let us suppose that the geometric mean of the $|d_k|$ which belong to the interval q_k are denoted by G_k and for the interval Q_k by $G_{k,1}$. Then the geometric mean G_q for the entire interval q will be

$$G_q=[G_0^{c_0}G_1^{c_1q_1}\prod_{k=2}^{n}G_k^{c_kq_k}G_{k,1}^{Q_k}]^{\frac{1}{q}},$$

where the quantities G_k and G_{k1} approach the limit $1/(2e)$ as k increases indefinitely.

The purpose of our investigation is now attained by the use of the following:

LEMMA: *If $j_1, j_2, \ldots,$ is any set of positive integers such that $j_1\leqq J_1$, $j_2\leqq J_2, \ldots,$ where $J_1, J_2, \ldots,$ is a given set of integers, and if $x_1, x_2, \ldots,$ is a set of positive quantities which approach x as a limit, then*

$$y_n=[\prod_{k=1}^{n}x_k^{j_k}]^{\frac{1}{j_1+\ldots+j_n}}$$

also approaches x as a limit as n increases indefinitely, provided there does not exist an $n=N$ such that if $m>N$ every $j_m=0$.

The proof of this lemma is simple and will be omitted. Since all of its conditions are satisfied by the expression for G_q, we have proved the following:

THEOREM: *If β is a given positive number, and p_n/n is a rational fraction such that $|p_n-n\beta|\leqq\frac{1}{2}$, and if $G_n=[\prod_{k=1}^{n}|p_k-k\beta|]^{\frac{1}{n}}$, then the limit of G_n as n*

increases indefinitely is zero if β *is rational, and is* $1/(2e)$ *if* β *is an irrational number which, when expressed as a simple continued fraction*

$$\beta = a_0 + \frac{1}{a_1 +}\,\frac{1}{a_2 +}\,\frac{1}{a_3 +}\cdots,$$

of which p_n/q_n *is the n-th convergent, satisfies the condition that the limit of* $\left(\frac{1}{a_{n+1}}\right)^{\frac{1}{q_n}} = 1.$

For every irrational number, β*, the limit of* G_{q_n-1} *as n increases indefinitely is* $1/(2e)$*, where e is the logarithmic base.*

If $E(k\beta)$ is the greatest integer contained in $k\beta$ and if $f_{k\beta} = k\beta - E(k\beta)$, it is readily seen that limit of $[\prod_{k=1}^{n} f_{k\beta}]^{1/n}$ is $\frac{1}{e}$, provided β satisfies the condition already stated.

In a manner, quite analogous to the preceding argument, it can be shown that the limit of the arithmetic mean of the quantities d_k (signs considered) is zero whether β is rational or irrational, provided that when β is rational and $d_k = \pm\frac{1}{2}$ is ambiguous such terms be taken alternately positive and negative. Likewise the limit the arithmetic mean of $|d_k|$ (signs discarded) is $\frac{1}{4}$ if β is any irrational number or any rational number which in its lowest terms has an even denominator; and is equal to $\frac{1}{4}\left(1 - \frac{1}{q^2}\right)$ if β is a rational number which, when expressed in its lowest terms, has an odd denominator q. The proof for the theorems with respect to the arithmetic means will not be necessary inasmuch as substantially equivalent theorems have been given by Sierpinski.*

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* See *Jahrbuch über die Fortschritte der Mathematik* (1909), p. 221; Hardy and Littlewood, *Proceedings of the Cambridge Congress of Mathematicians* (1912), Vol. I.